Hilbert C^*-modules (Lance)

Overview: Hilbert C^*-module is kind of a generalization of a Hilbert space:

- have a $\mathbb{C}$-valued inner product
- $\mathbb{C}$ acts on vectors

replace $\mathbb{C}$ by an arbitrary C^*-algebra

Def: Let A be a C^*-alg. An inner product A-module E is a complex vector space such that

$$\underset{\mathbb{C}}{\lambda}(\underset{E}{x}\,\underset{A}{a}) = (\lambda x)a = x(\lambda a)$$

with an A-valued inner-product:

$$\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$$

$$\langle y, xa\rangle = \langle y, x\rangle a$$

$$\langle y, x\rangle = \langle x, y\rangle^*$$

$$\langle x, x\rangle \geq 0 \;;\; \langle x, x\rangle = 0 \Rightarrow x = 0$$

prop (1.1) (Cauchy-Schwarz)

$$\underbrace{\langle y,x\rangle}_{\langle x,y\rangle^*}\langle x,y\rangle \leq \underbrace{\|\langle x,x\rangle\|_A}_{\text{norm on } A}\underbrace{\langle y,y\rangle}_{\langle y,y\rangle^*}$$

Can also make a norm on E:

$$\|x\|_E = \|\langle x,x\rangle\|_A^{1/2}$$

$$\overset{\text{prop 1.1}}{\Longrightarrow} \quad \|\langle x,y\rangle\|_A \leq \|x\|_E\|y\|_E$$

$x,y \in E$

$\langle x,y\rangle \in A$

Can also make an absolute value on E (A-valued)

$$|x|_E = \underbrace{\langle x,x\rangle^{1/2}}_{\text{positive element in } A}$$

$$\overset{\text{prop 1.1}}{\Longrightarrow} \quad |\langle x,y\rangle|_A \leq \|x\|_E\,|y|_E$$

$$\boxed{|a| = \sqrt{a^*a}}$$

Def: An inner-prod A-module which is complete with respect to its norm is called a <u>Hilbert A-module</u> (or a Hilbert C^*-module over A).

Ex:

- Hilbert $\mathbb{C}$-modules are Hilbert spaces
- a C^*-algebra A acting on itself is a Hilbert A-module:

$$x \cdot a = xa$$
$$\langle x, y \rangle = x^* y$$

- $M_{m \times n}(\mathbb{C})$ is a Hilbert $M_n(\mathbb{C})$-Modules

$$\underbrace{\Big[\quad\Big]}_{m \times n}\underbrace{\big[\;\big]}_{n \times n} = \underbrace{\Big[\quad\Big]}_{m \times n}$$

$$x \cdot a = xa$$
$$\langle x, y \rangle = x^* y$$
$$= \bar{x}^T y$$
$$\in M_n(\mathbb{C})$$

$$\bigoplus_{i=1}^{\infty} E_i = \left\{ (x_i)_{i=1}^{\infty} \;\middle|\; \sum_{i=1}^{\infty} \langle x_i, x_i \rangle \text{ converges in } A \right\}$$

each E_i is a Hilber A-module

if each $E_i = A$, then $H_A = \bigoplus_{i=1}^{\infty} A$.

e.g. Let $A = C([0,1])$, $F = \{ f \in A \mid f(0) = 0 \} \subseteq A$

By definition

$$F^{\perp} = \{ g \in A \mid \langle f, g \rangle = 0 \ \forall f \in F \}$$

$\langle f, g \rangle (t) = f^*(t) \cdot g(t)$ so $F^{\perp} = \{0\}$

then $F^{\perp\perp} = A$.

Def: A Hilbert A-module E is called full if the ideal

$$I = \operatorname{span} \{ \langle x, y \rangle_E \mid x, y \in E \} \subseteq A$$

is dense in A.

Maps between Hilbert C^*-modules

Let E, F be Hilbert A-modules.

Define $L(E,F)$ to be the set of all maps $t: E \to F$ for which there is a map $t^*: F \to E$ such that

$$\langle tx, y \rangle_F = \langle x, t^* y \rangle_E \qquad (x \in E, y \in F)$$

(these are called the adjointable operators)

$\Rightarrow$ t is A-linear and bounded (exercise)

Note: Bounded + Linear $\not\Rightarrow$ Adjointable

$A = C([0,1])$

$i : F \hookrightarrow A$ bounded and linear but has no adjoint.

Facts:

- $t \in L(E,F) \Rightarrow t^* \in L(F,E)$
- $s \in L(F,G) \Rightarrow st \in L(E,G)$
- $L(E) := L(E,E)$ is a C^*-algebra with adjoint

For $x \in E$, $y \in F$ define $\Theta_{x,y} : F \to E$ by

$$\Theta_{x,y}(z) = x \underbrace{\langle y, z \rangle_F}_{\in A} \qquad z \in F$$

$\Theta_{x,y} \in L(F,E)$ and $(\Theta_{x,y})^* = \Theta_{y,x}$. These are "rank 1" operators.

e.g. With A (unital) as a Hilbert A-module,

$\Theta_{1,1}$ is the identity operator.

Def: $K(F,E)$ is the closed linear subspace of $L(F,E)$ spanned by

$$\{\Theta_{x,y} \mid x \in E, y \in F\}$$

Note: • $K(E)$ is a closed two-sided ideal in $L(E)$

• $K(E)$ is not (in general) compact as operators on a Banach space.

ex: $K(A) \cong A$ as a A-mod

$$\theta_{a,b} \longmapsto ab^*$$

dense in $K(A)$ — dense in A

If A is unital, then $L(A) = K(A)$.

$$L(A) \ni t \longmapsto \theta_{t(1),1}$$

From A-linearity: $t(a) = t(1a) = t(1)a$

Riesz representation theorem for Hilbert A-module

Let E be a Hilbert A-module, fix $x \in E$

define $t_x : E \to A$ $\qquad \langle a,b\rangle = a^*b$

$$t_x y = \langle x,y\rangle$$

t_x has an adjoint:

$$\langle t_x y, a\rangle_A = \langle x,y\rangle^* a = \langle y,x\rangle a = \langle y, xa\rangle$$

$$t_x^* : A \to E \ : \ t_x^*(a) = xa \implies t_x \in L(E,A)$$

t_x is actually in $K(E,A)$

- $\theta_{a,z}(y) = a\langle z,y\rangle = \langle za^*, y\rangle = t_{za^*}(y)$
- $t_x \in K(E,A)$ whenever $x = za^*$
- use fact that $EA = \{xa \mid x \in E, a \in A\} \subseteq E$ is dense in E.
- every element of $K(E,A)$ is of this form (exercise).

If A is unital $K(E,A) = L(E,A)$ ...